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NEW INTEGRAL REPRESENTATIONS OF ANALYTICAL SOLUTIONS TO BOUNDARY-VALUE PROBLEMS OF NONSTATIONARY TRANSFER IN REGIONS WITH MOVING BOUNDARIES

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This article surveys the development of a Green function method in solving boundary-value problems of nonstationary transfer in a region with a moving boundary. For a uniform law of boundary displacement, a modification of the thermal-potential method is suggested that leads to analytical solutions in a new (simplest) integral form.

There is a wide range of problems in consideration of which one has to come up against the necessity of solving boundary-value problems for nonstationary transfer equations in regions of $[0, y(t)]$, $t \geq 0$ or $[y(t), \infty)$, $t \geq 0$ ($y(t)$ is a continuous function). Similar problems arise in theoretical studies of energy transfer processes associated with a change in the state of aggregation of a substance in the theory of strength, in the theory of dams, in soil mechanics, in the thermal study of oil beds, in electrodynamic problems, in filtration problems, in vibration theory, in the theory of zone refinement of materials, in the kinetic theory of crystal growth, in thermal mechanics when studying a thermal shock, etc. [1]. Mathematically, boundary-value problems of transfer in a region with moving boundaries differ in principle from classical ones. In view of the dependence of the region boundary on time, classical methods for equations of mathematical physics are inapplicable to this class of problems, since within the framework of these methods it is impossible to coordinate a solution of a heat conduction equation with motion of the region boundary. The natural way out of this situation is to develop new approaches or modify well-known ones for regions with moving boundaries.

1. Green Function Method for Parabolic Equations in Noncylindrical Regions. For regions with moving boundaries (noncylindrical regions) the Green function method is one of the most effective approaches. This method presupposes the consideration beforehand of a simpler model in determining the corresponding influence function (a Green function) and allows one to obtain an integral representation of analytical solutions to an extensive class of unsteady transfer problems, depending on the inhomogeneities in the initial statement of the problem. However, for noncylindrical regions specific features appear that are peculiar to the presence of moving boundaries. At first we dwell briefly on the method indicated for cylindrical (classical) regions.

Let D be a finite or partially bounded convex region of change of $M(x, y, z)$; S is the piecewise smooth surface that bounds region D; n is the outward normal to S; $\Omega = (M \in D, t > 0)$ is a cylindrical region in the phase space (x, y, z, t) with the base D at $t = 0$; $T(M, t)$ is a temperature function that satisfies the following conditions of the problem:

$$\frac{\partial T}{\partial t} = a\Delta T(M, t) + f(M, t), \quad M \in D, \quad t > 0; \tag{1}$$

$$T(M, t)|_{t=0} = \Phi_0(M), \quad M \in \overline{D}; \tag{2}$$

$$\beta_1 \frac{\partial T(M, t)}{\partial n} + \beta_2 T(M, t) = \varphi(M, t), \quad M \in S, \quad t \geq 0. \tag{3}$$

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Here

$$f(M, t) \in C^0(\overline{\Omega})\,; \quad \Phi_0(M) \in C^1(\overline{\Omega})\,; \quad \varphi(M, t) \in C^0(S \times t \geq 0)\,;$$

$$\overline{D} = D + S\,, \quad \overline{\Omega} = (M \in \overline{D},\, t \geq 0)\,.$$

The desired solution is

$$T(M, t) \in C^2(\Omega) \cap C^0(\overline{\Omega})\,; \quad \mathrm{grad}_M\, T(M, t) \in C^0(\overline{\Omega})\,; \quad \beta_1^2 + \beta_2^2 > 0\,.$$

By virtue of the superposition principle for linear transfer problems, the integral representation for $T(M, t)$ can be written in the form of [1]

$$T(M, t) = \iiint\limits_D \Phi_0(P)\, G(M, P, t, \tau)|_{\tau=0}\, dV_P + a \int_0^t \iint\limits_S \left(G \frac{\partial T}{\partial n_P} - \right.$$

$$\left. - T \frac{\partial G}{\partial n_P} \right)_{P \in S} d\tau d\sigma_P + \int_0^t \iiint\limits_D f(P, \tau)\, G(M, t, P, \tau)\, d\tau dV_P\,, \tag{4}$$

if the corresponding Green function $G(M, t, P, r)$ is known for the given region as a solution of a simpler problem for homogeneous equation (1) with homogeneous boundary conditions of the same forms as in Eq. (3):

$$\frac{\partial G}{\partial t} = a\Delta_M G(M, t, P, \tau)\,, \quad M \in D\,, \quad t > \tau\,; \tag{5}$$

$$G(M, t, P, \tau)|_{t=\tau} = \delta(M, P)\,, \quad M \in D\,, \quad P \in D\,; \tag{6}$$

$$\beta_1 \frac{\partial G(M, t, P, \tau)}{\partial n_M} + \beta_2 G(M, t, P, \tau) = 0\,, \quad M \in S\,, \quad t > \tau\,. \tag{7}$$

Here $\delta(M, P)$ is the Dirac delta function. If we consider $G(M, t, P, \tau)$ a function of the point P and time τ, then in order to find G, we must solve an equivalent problem for the conjugate of Eq. (5):

$$\frac{\partial G}{\partial \tau} + a\Delta_P G(M, t, P, \tau) = 0\,, \quad P \in D\,, \quad \tau < t\,; \tag{8}$$

$$G(M, t, P, \tau)|_{\tau=t} = \delta(P, M)\,, \quad P \in D\,, \quad M \in D\,; \tag{9}$$

$$\beta_1 \frac{\partial G(M, t, P, \tau)}{\partial n_P} + \beta_2\, G(M, t, P, \tau) = 0\,, \quad P \in S\,, \quad \tau < t\,. \tag{10}$$

If the region D is bounded, the Green function G has the form [1]

$$G(M, t, P, \tau) = G(M, t - \tau, P,) = \sum_{n=1}^{\infty} \frac{\Psi_n(M)\, \Psi_n(P)}{\|\Psi_n\|^2} \exp\left[-(\sqrt{a}\, \gamma_n)^2 (t - \tau)\right],$$

where $\Psi_n(M)$ and γ_n^2 are the eigenfunctions and eigenvalues of the homogeneous problem corresponding to Eqs. (1)-(3):

$$\Delta\Psi(M) + \gamma^2 \Psi(M) = 0\,, \quad M \in D\,;$$

$$\beta_1 \frac{\partial \Psi(M)}{\partial n} + \beta_2 \Psi(M) = 0, \quad M \in S.$$

Here $\|\Psi_n\|^2$ is the norm square of the eigenfunctions

$$\| \Psi_n \|^2 = \int\int_D\int \Psi_n^2(M)\, dV_M.$$

Now let Ω_t be a noncylindrical region, i.e., the cross section of Ω_t with the plane-characteristic t = const $\geq t_0 > 0$ is the region D_t with the boundary S_t depending on time t. We find changes in the statements of boundary-value problems (5)-(7), (8)-(10) with respect to the Green function $G(M, t, P, \tau)$ in the variables (M, t) and (P, τ) (for the cylindrical regions, the formulation of the boundary conditions remains unchanged; equation (5) is replaced by conjugate equation (8)).

We consider the region $\overline{\Omega}_t = (0 \leq x \leq y(t),\ t \geq 0)$, where $y(t)$ is a continuously differentiable function, and $T(x, t)$ is a solution of the problem:

$$\frac{\partial T}{\partial t} = a \frac{\partial^2 T}{\partial x^2} + f(x, t), \quad 0 < x < y(t), \quad t > 0; \tag{11}$$

$$T(x, t)|_{t=0} = \Phi_0(x), \quad 0 \leq x \leq y(0), \quad y(0) \geq 0, \tag{12}$$

with boundary conditions of the first kind

$$\begin{aligned} T(x, t)|_{x=0} &= \varphi_1(t), \quad t \geq 0; \\ T(x, t)|_{x=y(t)} &= \varphi_2(t), \quad t \geq 0; \end{aligned} \tag{13}$$

the second kind

$$\begin{aligned} \left.\frac{\partial T(x, t)}{\partial x}\right|_{x=0} &= \varphi_1(t), \quad t \geq 0; \\ \left.\frac{\partial T(x, t)}{\partial x}\right|_{x=y(t)} &= \varphi_2(t), \quad t \geq 0, \end{aligned} \tag{14}$$

or the third kind

$$\begin{aligned} \left.\frac{\partial T(x, t)}{\partial x}\right|_{x=0} &= h_1 \left[T(x, t)|_{x=0} - \varphi_1(t) \right], \quad t \geq 0; \\ \left.\frac{\partial T(x, t)}{\partial x}\right|_{x=y(t)} &= -h_2 \left[T(x, t)|_{x=y(t)} - \varphi_2(t) \right], \quad t \geq 0. \end{aligned} \tag{15}$$

Characteristic of the given problem is the presence of boundaries moving in time and, consequently, the fact that the function $G(x, t, x', \tau)$ in view of its physical sense (a heat pulse of power $Q = c\rho$ [2]) depends not on the difference $(t - \tau)$, but on t and τ, since not only the action time $(t - \tau)$, but also the pulse onset moment τ will be determining factors. We present the function $G(x, t, x', \tau)$ in the form of [3]

$$G(x, t, x, \tau) = \frac{1}{2\sqrt{\pi a (t - \tau)}} \exp\left[-\frac{(x - x')^2}{4a(t - \tau)} \right] + q(x, t, x', \tau) = G_0 + q, \quad \tau < t, \tag{16}$$

where G_0 is a fundamental solution of homogeneous equation (11). The function G_0 satisfies homogeneous equation (11) with respect to the variables (x, t) and its conjugate equation with respect to the variables (x', τ). We select a function q that is a regular component of Green function (16) such that the equation $q'_\tau = -aq''_{x'x'}$, $\tau < t$ and the initial condition $q(x, t, x', \tau = t) = 0$ are satisfied. For the function $G(x, t, x', \tau)$ with respect to the variables (x', τ) we have (in conformity with Eqs. (8) and (9)):

$$\frac{\partial G}{\partial \tau} = - a \frac{\partial^2 G}{\partial x'^2}, \quad 0 < x' < y(\tau), \quad \tau < t; \tag{17}$$

$$G(x, t, x', \tau)|_{\tau=t} = \delta(x' - x), \quad 0 < x' < y(t). \tag{18}$$

Now we consider the equality

$$\frac{\partial}{\partial \tau} [T(x', \tau) G(x, t, x', \tau)] = G \frac{\partial T}{\partial \tau} + T \frac{\partial G}{\partial \tau} = a \left(G \frac{\partial^2 T}{\partial x'^2} - T \frac{\partial^2 G}{\partial x'^2} \right) + Gf(x', \tau). \tag{19}$$

Integration of Eq. (19) with respect to $x' \in [0, y(\tau)]$ yields

$$\int_0^{y(\tau)} \frac{\partial}{\partial \tau} (TG)\, dx' = a \left(G \frac{\partial T}{\partial x'} - T \frac{\partial G}{\partial x'} \right)_{x'=0}^{x'=y(\tau)} + \int_0^{y(\tau)} f(x', \tau) G(x, t, x', \tau)\, dx'. \tag{20}$$

Relation (20) is valid for all $\tau < t$ and therefore can be integrated with respect to τ for $0 < \tau < t - \varepsilon$, where $\varepsilon > 0$ is an arbitrarily small number (when $0 < \tau < t - \varepsilon$, the integrands in Eq. (20) are regular enough, since the singularity of the function G at the point $x' = x$ with $\tau = t$ is eliminated). We obtain:

$$\int_0^{t-\varepsilon} d\tau \int_0^{y(\tau)} \frac{\partial}{\partial \tau} (TG)\, dx' = a \int_0^{t-\varepsilon} \left(G \frac{\partial T}{\partial x'} - T \frac{\partial G}{\partial x'} \right)_{x'=y(\tau)} d\tau -$$

$$- a \int_0^{t-\varepsilon} \left(G \frac{\partial T}{\partial x'} - T \frac{\partial G}{\partial x'} \right)_{x'=0} d\tau + \int_0^{t-\varepsilon} d\tau \int_0^{y(\tau)} f(x', \tau) G(x, t, x', \tau)\, dx'. \tag{21}$$

Removing from the left side the operator $\partial/\partial\tau$ after the definite-integral sign and passing to the limit as $\varepsilon \to 0$, which gives from Eq.(18)

$$\lim_{\varepsilon \to 0} \int_0^{y(t-\varepsilon)} T(x', t - \varepsilon) G(x, t, x', \tau)|_{\tau=t-\varepsilon}\, dx' = \int_0^{y(t)} T(x', t) \delta(x' - x)\, dx' = T(x, t),$$

we obtain an integral formula for arbitrary solutions of Eq. (11) in a region with a moving boundary

$$T(x, t) = \int_0^{y(0)} [T(x', \tau) G(x, t, x', \tau)]_{\tau=0}\, dx' - a \int_0^t \left(G \frac{\partial T}{\partial x'} - T \frac{\partial G}{\partial x'} \right)_{x'=0} d\tau +$$

$$+ a \int_0^t \left[G \frac{\partial T}{\partial x'} - T \left(\frac{\partial G}{\partial x'} - \frac{1}{a} \frac{dy}{d\tau} G \right) \right]_{x'=y(\tau)} d\tau + \int_0^t \int_0^{y(\tau)} f(x', \tau) G(x, t, x', \tau)\, d\tau dx'. \tag{22}$$

Suppose that in the statement of the initial problem for Eq. (11) we assign boundary conditions of the first kind (13), the second kind (14), or the third kind (15) (or mixed boundary conditions, thus leaving the essence of the matter unchanged). By choosing a function $G(x, t, x', \tau)$ that satisfies the following boundary conditions:

a) for the first boundary-value problem

$$G(x, t, x', \tau)|_{x'=0} = 0, \quad G(x, t, x', \tau)|_{x'=y(\tau)} = 0, \quad \tau < t; \tag{23}$$

b) for the second boundary-value problem

$$\left.\frac{\partial G}{\partial x'}\right|_{x'=0} = 0, \quad \left(\frac{\partial G}{\partial x'} - \frac{1}{a}\frac{dy}{d\tau} G\right)_{x'=y(\tau)} = 0, \quad \tau < t; \tag{24}$$

c) for the third boundary-value problem

$$\left(\frac{\partial G}{\partial x'} - h_1 G\right)_{x'=0} = 0, \quad \left[\frac{\partial G}{\partial x'} + \left(h_2 - \frac{1}{a}\frac{dy}{d\tau}\right) G\right]_{x'=y(\tau)} = 0, \quad \tau < t, \tag{25}$$

hen from Eq. (22) we obtain the desired integral representations for analytical solutions of $T(x, t)$ in the form of:

a) the first boundary-value problem

$$T(x, t) = \int_0^{y(0)} [T(x', \tau) G(x, t, x', \tau)]_{\tau=0} dx' + a \int_0^t \left[T(x', \tau) \frac{\partial G}{\partial x'}\right]_{x'=0} d\tau -$$

$$- a \int_0^t \left[T(x', \tau) \frac{\partial G}{\partial x'}\right]_{x'=y(\tau)} d\tau + \int_0^t \int_0^{y(\tau)} f(x', \tau) G(x, t, x', \tau) d\tau dx'; \tag{26}$$

b) the second boundary-value problem

$$T(x, t) = \int_0^{y(0)} \left[T(x', \tau) G(x, t, x', \tau)\right]_{\tau=0} dx' - a \int_0^t \left[\frac{\partial T(x', \tau)}{\partial x'} G\right]_{x'=0} d\tau +$$

$$+ a \int_0^t \left[\frac{\partial T(x', \tau)}{\partial x'} G\right]_{x'=y(\tau)} d\tau + \int_0^t \int_0^{y(\tau)} f(x', \tau) G(x, t, x', \tau) d\tau dx'; \tag{27}$$

c) the third boundary-value problem

$$T(x, t) = \int_0^{y(0)} [T(x', \tau) G(x, t, x', \tau)]_{\tau=0} dx' - a \int_0^t \left[\left(\frac{\partial T(x', \tau)}{\partial x'} - h_1 T(x', \tau)\right) G\right]_{x'=0} d\tau +$$

$$+ a \int_0^t \left[\left(\frac{\partial T(x', \tau)}{\partial x'} + h_2 T(x', \tau)\right) G\right]_{x'=y(\tau)} d\tau + \int_0^t \int_0^{y(\tau)} f(x', \tau) G(x, t, x', \tau) d\tau dx'. \tag{28}$$

As is seen from Eqs. (24) and (25), the construction of the Green function $G(x, t, x', \tau)$ for the second and third boundary-value problems (17), (18), (24) and (17), (18), (25) on the basis of the statement of the problem with respect to the variable (x', τ) involves serious difficulties and for the majority of boundary motion laws is technically impracticable. If we proceed from the problem statement with respect to the variables (x, t), then the situation is substantially simplified. Actually, let us consider the function $\overline{G}(x, t, x', \tau)$ determined by the following condition:

$$\frac{\partial \overline{G}}{\partial t} = a \frac{\partial^2 \overline{G}}{\partial x^2}, \quad 0 < x < y(t), \quad t > \tau; \tag{29}$$

$$\overline{G}(x, t, x', \tau)|_{t=\tau} = \delta(x - x'), \quad 0 < x < y(\tau): \tag{30}$$

a) for the first boundary-value problem

$$\bar{G}|_{x=0} = 0\,, \quad t > \tau\,;$$
$$\bar{G}|_{x=y(t)} = 0\,, \quad t > \tau\,; \tag{31}$$

b) for the second boundary-value problem

$$\left.\frac{\partial \bar{G}}{\partial x}\right|_{x=0} = 0\,, \quad t > \tau\,;$$
$$\left.\frac{\partial \bar{G}}{\partial x}\right|_{x=y(t)} = 0\,, \quad t > \tau\,; \tag{32}$$

c) for the third boundary-value problem

$$\left(\frac{\partial \bar{G}}{\partial x} - h_1 \bar{G}\right)_{x=0} = 0\,, \quad t > \tau\,;$$
$$\left(\frac{\partial \bar{G}}{\partial x} + h_2 \bar{G}\right)_{x=y(t)} = 0\,, \quad t > \tau\,. \tag{33}$$

We will show that $G(x, t, x', \tau) = \bar{G}(x, t, x', \tau)$.

First we integrate the expression

$$\frac{\partial}{\partial \theta}\,[\bar{G}\,(x'', \theta, x', \tau)\, G\,(x, t, x'', \theta)\,] = a\left(\bar{G}\,\frac{\partial^2 \bar{G}}{\partial x''^2} - \bar{G}\,\frac{\partial^2 G}{\partial x''^2}\right)$$

with respect to x'' in the interval $[0, y(\theta)\,]$, where $t > \theta > \tau$. Taking into account the boundary conditions for $\bar{G}$ and G (for all the above-indicated types), we obtain:

$$\frac{\partial}{\partial \theta}\int_0^{y(\theta)} \bar{G}\, G dx'' = 0\,. \tag{34}$$

Then we integrate Eq. (34) with respect to θ over the interval $[\theta, t - \varepsilon\,]$, where $\varepsilon > 0$ is an arbitrarily small number; repeating the previous reasoning (just as in deriving relation (22)), we find (with $\varepsilon \to 0$) that

$$\int_0^{y(\theta)} \bar{G}\,(x'', \theta, x', \tau)\, G\,(x, t, x'', \theta)\, dx'' = \bar{G}\,(x, t, x', \tau)\,. \tag{35}$$

On the other hand, integrating Eq. (34) with respect to θ over the interval $[\tau + \varepsilon, \theta\,]$ and letting $\varepsilon \to 0$, we come to the expression:

$$\int_0^{y(\theta)} G\,(x, t, x'', \theta)\, \bar{G}\,(x'', \theta, x', \tau)\, dx'' = \bar{G}\,(x, t, x', \tau)\,. \tag{36}$$

A comparison between Eqs. (35) and (36) shows that $G(x, t, x', \tau) = \bar{G}((x, t, x', \tau)$.

Thus, the function $G(x, t\ x', \tau)$ can be found as a solution of equivalent problems for Eqs. (17) and (29) with the above boundary conditions (initial and boundary), and in regions with moving boundaries the equivalence is not retained in the boundary conditions in statements of problems with respect to (x, t) and (x', τ), unlike in classical cylindrical regions. This circumstance caused errors in a number of works on the construction of Green functions for boundary-value problems of unsteady heat transfer in regions with moving boundaries; this fact prompted the author to prepare the present publication.

2. Method of Integral Equations in Constructing a Green Function in a Region with a Uniformly Moving Boundary. Any case of finding a Green function of a corresponding boundary-value problem for a region with a moving boundary is of exceptional importance, since it contains extensive information about analytical solutions on the basis of integral relations (26)-(28).

Let us next consider a modification of the method of thermal potentials (method of integral equations) for constructing Green functions in a region with a uniformly moving boundary $[0, l + vt]$, $t \geq 0$ that is of interest for numerous applications, such as thermomechanics [4] and physics of strength [5]. Below we propose an approach that leads to analytical solutions in a new (simplest) integral form different from those known earlier for this case obtained by a different method. For brevity we consider the first boundary-value problem:

$$\frac{\partial G}{\partial t} = a\frac{\partial^2 G}{\partial x^2}, \quad 0 < x < l + vt, \quad t > \tau; \tag{37}$$

$$G|_{t=\tau} = \delta(x - x'), \quad 0 < x < l + v\tau; \quad G|_{x=0} = G|_{x=l+vt} = 0, \quad t > \tau. \tag{38}$$

Now we pass to the function $q(x, t\ x', \tau)$ by means of Eq. (16), whence, using Eqs. (37) and (38), we have:

$$\frac{\partial q}{\partial t'} = a\frac{\partial^2 q}{\partial x^2}, \quad 0 < x < l_0 + vt', \quad t' > 0; \tag{39}$$

$$q|_{t'=0} = 0, \quad 0 < x < l_0; \tag{40}$$

$$q|_{x=0} = -\frac{1}{2\sqrt{\pi a t'}}\exp\left(-\frac{x'^2}{4at'}\right), \quad t' > 0; \tag{41}$$

$$q|_{x=l_0+vt'} = -\frac{1}{2\sqrt{\pi a t'}}\exp\left[-\frac{(l_0 + vt' - x')^2}{4at'}\right], \quad t' > 0. \tag{42}$$

Here $t' = t - \tau$; $l_0 = l + vt$. A solution of problem (39)-(42) is sought in the form of a sum of potentials

$$q(x, t, x', \tau) = \frac{\sqrt{a}}{2\sqrt{\pi}}\int_0^{t'}\frac{\Psi_1(\tau)}{\sqrt{t' - \tau}}\exp\left[-\frac{x^2}{4a(t' - \tau)}\right]d\tau +$$

$$+\frac{\sqrt{a}}{2\sqrt{\pi}}\int_0^{t'}\frac{\Psi_2(\tau)}{\sqrt{t' - \tau}}\exp\left[-\frac{(x - l_0 - v\tau)^2}{4a(t' - \tau)}\right]d\tau, \tag{43}$$

where $\Psi_1(t')$and $\Psi_2(t')$ are the unknown potential densities to be determined. Expression (43) will be written in the space of Laplace transform

$$\bar{q} = \frac{\sqrt{a}}{2\sqrt{p}}\exp\left(-\frac{x}{\sqrt{a}}\sqrt{p}\right)\bar{\Psi}_1(p) + \frac{\sqrt{a}}{2\sqrt{p}}\exp\left(-\frac{l_0 - x}{\sqrt{a}}\sqrt{p}\right)\bar{\Psi}_3(\sqrt{p} + \gamma)^2, \tag{44}$$

where $\gamma = v/2\sqrt{a}$; $\Psi_3(t') = \Psi_2(t')\exp(\gamma^2 t')$. Thus, hereafter to find the original q from Eq. (44), it is necessary to seek transforms of the unknown densities in Eq. (43) relative to $\bar{\Psi}_1(p)$ and $\bar{\Psi}_3[(\sqrt{p} + \gamma)^2]$.

Satisfying boundary conditions (41)-(42) in Eq. (43), we obtain a system of Volterra integral equations in $\Psi_1(t')$ and $\Psi_3(t')$:

$$\frac{\sqrt{a}}{2\sqrt{\pi}}\int_0^{t'}\frac{\Psi_1(\tau)}{\sqrt{t'-\tau}}\,d\tau+\frac{\sqrt{a}}{2\sqrt{\pi}}\int_0^{t'}\frac{\Psi_3(\tau)}{\sqrt{t'-\tau}}\exp\left[-\frac{l_0^2+2l_0v\tau+v^2\tau t'}{4a(t'-\tau)}\right]d\tau=-\frac{1}{2\sqrt{\pi a t'}}\exp\left(-\frac{x'^2}{4at'}\right);\quad(45)$$

$$\frac{\sqrt{a}}{2\sqrt{\pi}}\int_0^{t'}\frac{\Psi_1(\tau)}{\sqrt{t'-\tau}}\exp\left[-\frac{l_0^2+2l_0vt'+v^2\tau t'}{4a(t'-\tau)}\right]d\tau+\frac{\sqrt{a}}{2\sqrt{\pi}}\int_0^{t'}\frac{\Psi_3(\tau)}{\sqrt{t'-\tau}}\,d\tau=$$

$$=\frac{1}{2\sqrt{\pi a t'}}\exp\left[-\frac{(l_0-x')^2}{4at'}-\frac{(l_0-x')\,v}{2a}\right].\quad(46)$$

In the transform space the system of integral equations takes the form:

$$\overline{\Psi}_1(p)+\exp\left(-\frac{l_0}{\sqrt{a}}\sqrt{p}\right)\overline{\Psi}_3(\sqrt{p}+\gamma)^2=-\frac{1}{a}\exp\left(-\frac{x}{\sqrt{a}}\sqrt{p}\right),$$

$$\exp\left[-\frac{l_0}{\sqrt{a}}(\sqrt{p}+\gamma)\right]\overline{\Psi}_1(\sqrt{p}+\gamma)^2+\overline{\Psi}_3(p)=-\frac{1}{a}\exp\left[-\frac{l_0-x'}{\sqrt{a}}(\sqrt{p}+\gamma)\right].\quad(47)$$

Elimination first of $\overline{\Psi}_3(p)$ and then of $\overline{\Psi}_1(p)$ from the system of functional equations (47) yields:

$$\exp\left[-\frac{l_0}{\sqrt{a}}(p+2\gamma)\right]\overline{\Psi}_1(\sqrt{p}+2\gamma)^2-\exp\left(\frac{l_0}{\sqrt{a}}\sqrt{p}\right)\overline{\Psi}_1(p)=$$

$$=-\frac{1}{a}\left\{\exp\left[-\frac{l_0-x'}{\sqrt{a}}(\sqrt{p}+2\gamma)\right]-\exp\left(-\frac{l_0-x'}{\sqrt{a}}\sqrt{p}\right)\right\};\quad(48)$$

$$\exp\left[-\frac{l_0}{\sqrt{a}}(\sqrt{p}+\gamma)\right]\overline{\Psi}_3(\sqrt{p}+2\gamma)^2-\exp\left[\frac{l_0}{\sqrt{a}}(\sqrt{p}+\gamma)\right]\overline{\Psi}_3(p)=$$

$$=-\frac{1}{a}\left\{\exp\left[-\frac{x'}{\sqrt{a}}(\sqrt{p}+\gamma)\right]-\exp\left[\frac{x'}{\sqrt{a}}(\sqrt{p}+\gamma)\right]\right\}.\quad(49)$$

We let:

$$\overline{\Psi}_1(p^2)=\overline{A}_1(p);\quad\overline{\Psi}_3(p^2)=\overline{A}_3(p)\quad(50)$$

and rewrite Eqs. (48) and (49) in the following form:

$$\exp\left(-\frac{2l_0}{\sqrt{a}}\gamma\right)\overline{A}_1(p+2\gamma)-\exp\left(\frac{2l_0}{\sqrt{a}}p\right)\overline{A}_1(p)=$$

$$=-\frac{1}{a}\left\{\exp\left[\frac{x'}{\sqrt{a}}p-\frac{2(l_0-x')}{\sqrt{a}}\gamma\right]-\exp\left(\frac{2l_0-x'}{\sqrt{a}}p\right)\right\};\quad(51)$$

$$\exp\left(-\frac{2l_0}{\sqrt{a}}\gamma\right)\overline{A}_3(p+2\gamma)-\exp\left(\frac{2l_0}{\sqrt{a}}p\right)\overline{A}_3(p)=$$

$$= -\frac{1}{a}\left[\exp\left(\frac{l_0 - x'}{\sqrt{a}}p - \frac{l_0 + x'}{\sqrt{a}}\gamma\right) - \exp\left(\frac{l_0 + x'}{\sqrt{a}}p - \frac{l_0 - x'}{\sqrt{a}}\gamma\right)\right]. \tag{52}$$

By means of substitution

$$\overline{A}_i(p) = \exp\left(\frac{l_0}{2\sqrt{a}\,\gamma}p^2\right)\overline{B}_i(p) \quad (i = 1;\ 3) \tag{53}$$

equations (51) and (52) are reduced to equations with constant coefficients:

$$\overline{B}_1(p + 2\gamma) - \overline{B}_1(p) = -\frac{1}{a}\exp\left(-\frac{l_0}{2\sqrt{a}\,\gamma}p^2\right) \times$$

$$\times\left\{\exp\left[-\frac{2l_0 - x'}{\sqrt{a}}p - \frac{2(l_0 - x')}{\sqrt{a}}\gamma\right] - \exp\left(-\frac{x'}{\sqrt{a}}p\right)\right\}; \tag{54}$$

$$\overline{B}_3(p + 2\gamma) - \overline{B}_3(p) = -\frac{1}{a}\exp\left(-\frac{l_0}{2\sqrt{a}\,\gamma}p^2\right) \times$$

$$\times\left\{\exp\left[-\frac{l_0 + x'}{\sqrt{a}}(p + \gamma)\right] - \exp\left[-\frac{l_0 - x'}{\sqrt{a}}(p + \gamma)\right]\right\}. \tag{55}$$

It can be directly verified that the function $\overline{F}(p) = -\sum_{n=0}^{\infty}\overline{C}(p + bn)$ [6] is a desired particular solution of the functional equation $\overline{F}(p + b - \overline{F}(p) = \overline{C}(p)$, provided that this series converges. But a series of the type of $\sum_{n=0}^{\infty}\exp$ $[-(l_0/2\sqrt{a}\gamma)(p + 2\gamma n)^2 - \overline{d}(n, p)]$ (where $\overline{d}(n, p) > 0$ are linear functions with respect to n), to which the solutions of Eqs. (54) and (55) are reduced, converges because $(l_0/2\sqrt{a}\gamma) > 0$. Thus, the following functions will be solutions of Eqs. (54) and (55):

$$\overline{B}_1(p) = \frac{1}{a}\sum_{n=0}^{\infty}\exp\left[-\frac{l_0}{2\sqrt{a}\,\gamma}(p + 2\gamma n)^2\right] \times$$

$$\times\left\{\exp\left[-\frac{2l_0 - x'}{\sqrt{a}}(p + 2\gamma n) - \frac{2(l_0 - x')}{\sqrt{a}}\gamma\right] - \exp\left(-\frac{x'}{\sqrt{a}}(p + 2\gamma n)\right)\right\}; \tag{56}$$

$$\overline{B}_3(p) = \frac{1}{a}\sum_{n=0}^{\infty}\exp\left[-\frac{l_0}{2\sqrt{a}\,\gamma}(p + 2\gamma n)^2\right] \times$$

$$\times\left\{\exp\left[-\frac{l_0 + x'}{\sqrt{a}}(p + (2n + 1)\gamma)\right] - \exp\left[\frac{l_0 - x'}{\sqrt{a}}(p + (2n + 1)\gamma)\right]\right\}. \tag{57}$$

Knowing $\overline{B}_i(p)$, we first find $\overline{A}_i(p)$ from Eq. (53) and then, with allowance for Eq. (50), the desired potential densities relative to the transforms that enter into Eq. (44):

$$\overline{\Psi}_1(p) = \frac{1}{a}\sum_{n=0}^{\infty}\exp\left\{-\frac{2\gamma}{\sqrt{a}}(n + 1)\,|(n + 1)\,l_0 - x'\,| - \frac{2l_0(n + 1) - x'}{\sqrt{a}}\sqrt{p}\right\} -$$

$$-\frac{1}{a}\sum_{n=0}^{\infty}\exp\left(-\frac{2l_0\gamma}{\sqrt{a}}n^2-\frac{2\gamma x'}{\sqrt{a}}n-\frac{2l_0n+x'}{\sqrt{a}}\sqrt{p}\right), \tag{58}$$

$$\Psi_3(\sqrt{p}+\gamma)^2=\frac{1}{a}\sum_{n=0}^{\infty}\exp\left[-\frac{2l_0\gamma}{\sqrt{a}}(n+1)^2-\frac{2\gamma x'}{\sqrt{a}}(n+1)-\frac{(2n+1)\,l_0+x'}{\sqrt{a}}\sqrt{p}\right]-$$

$$-\frac{1}{a}\sum_{n=0}^{\infty}\exp\left\{-\frac{2\gamma}{\sqrt{a}}(n+1)\,[(n+1)\,l_0-x']-\frac{(2n+1)\,l_0-x'}{\sqrt{a}}\sqrt{p}\right\}. \tag{59}$$

Substituting Eqs. (58) and (59) into Eq. (44) and passing to the space of original functions, we determine, with allowance for Eq. (16), the desired Green function $G(x, t, x', \tau)$:

$$G(x,t,x',\tau)=\frac{1}{2\sqrt{\pi a(t-\tau)}}\sum_{n=-\infty}^{n=+\infty}\exp\left(-\frac{2l_0\gamma}{\sqrt{a}}n^2-\frac{2\gamma x'}{\sqrt{a}}n\right)\times$$

$$\times\left\{\exp\left[-\frac{(2l_0n+x'-x)^2}{4a(t-\tau)}\right]-\exp\left[-\frac{(2l_0n+x'+x)^2}{4a(t-\tau)}\right]\right\}, \tag{60}$$

where $l_0 = l + v\tau$; $\gamma = v/2\sqrt{a}$. Analyzing the expression obtained, we combine the exponentials that contain n^2 in the exponent. Then under the sign of the series the factor $\exp[-l_0n^2(l + vt)/a(t - \tau)]$ appears. From this it follows that series (60) converges also for negative values of v provided that $\tau < t < -(l/v)$. But when $t = -l/v$ ($v < 0$), the prescribed region disappears. Consequently, expression (60) can be used for any values of v.

Now, using Eq. (26), it is possible to write an integral representation of the analytical solution of first boundary-value problem (11)-(13) in terms of Green function (60):

$$T(x,t)=\int_0^l\Phi_0(x')\,G(x,t,x',0)\,dx'+a\int_0^t\varphi_1(\tau)\left.\frac{\partial G}{\partial x'}\right|_{x'=0}d\tau-$$

$$-a\int_0^t\varphi_2(\tau)\left.\frac{\partial G}{\partial x'}\right|_{x'=l+v\tau}d\tau+\int_0^t\int_0^{l+v\tau}f(x',\tau)\,G(x,t,x',\tau)\,d\tau dx'. \tag{61}$$

When $\Phi_0(x) = 0$, $f(x, t) = 0$, expression (61) takes the form

$$T(x,t)=\frac{1}{2\sqrt{a\pi}}\sum_{n=-\infty}^{n=+\infty}\int_0^t\frac{x+2n(l+v\tau)}{(t-\tau)^{3/2}}\varphi_1(\tau)\exp\left\{-\frac{v(l+v\tau)\,n^2}{a}-\right.$$

$$\left.-\frac{[x+2n(l+v\tau)]^2}{4a(t-\tau)}\right\}d\tau-\frac{1}{2\sqrt{a\pi}}\sum_{n=-\infty}^{n=+\infty}\int_0^t\frac{[x+(2n+1)(l+v\tau)]}{(t-\tau)^{3/2}}\varphi_2(\tau)\times$$

$$\times\exp\left\{-\frac{v(l+v\tau)}{a}n(n+1)-\frac{[x+(2n+1)(l+v\tau)]^2}{4a(t-\tau)}\right\}d\tau, \tag{62}$$

thus representing an integral relation of the new form for the first boundary-value problem of nonstationary thermal conductivity in the region of $[0, l + vt]$, $t \geq 0$. Expression (62) has an interesting continuation. We pass in Eq. (62) to the space of transforms (following Laplace)

$$\overline{T}(x,p)=\overline{\varphi}_1(p)\exp\left(-\frac{x}{\sqrt{a}}\sqrt{p}\right)+\frac{1}{\sqrt{p}}\sum_{n=1}^{\infty}\exp\left(-\frac{2l\gamma}{\sqrt{a}}n^2\right)(\sqrt{p}+2\gamma n)\times$$

$$\times \left[\exp\left(-\frac{2ln + x}{\sqrt{a}}\sqrt{p}\right) - \exp\left(-\frac{2ln - x}{\sqrt{a}}\sqrt{p}\right)\right] \bar{\varphi}_1 \left[(\sqrt{p} + 2\gamma n)^2\right] +$$

$$+ \frac{1}{\sqrt{p}} \sum_{n=0}^{\infty} \exp\left[-\frac{2l\gamma}{\sqrt{a}} n (n + 1)\right] |\sqrt{p} + (2n + 1)\gamma| \times$$

$$\times \left\{\exp\left[-\frac{(2n + 1) l - x}{\sqrt{a}}\sqrt{p}\right] - \exp\left[-\frac{(2n + 1) l + x}{\sqrt{a}}\sqrt{p}\right]\right\} \bar{\varphi}_3 (\sqrt{p} + (2n + 1)\gamma)^2, \tag{63}$$

where $\varphi_3(t) = \varphi_2(t) \exp(\gamma^2 t)$. Expression (63) can be a working formula for writing analytical solutions (in transforms) of first boundary-value problem (11)-(13) (when $f = \Phi_0 = 0$) in the region of $[0, l + vt]$, $t \ge 0$ for a wide class of boundary functions (homogeneous, impulse, pulsating, periodic, etc.). From this expression it is also possible to obtain immediately a representation for the Green function, if one takes into account that in this case $\varphi_1(t)$ and $\varphi_2(t)$ are well-known functions (41) and (42). The practical usefulness of expression (63) lies in the fact that with the prescribed boundary functions the direct passage to the original functions in Eq. (63) eliminates prolonged intermediate calculations, as for instance, in passage from Eq. (61) to Eq. (62) (it is clear that relation (63) could be also obtained immediately by the previous reasoning on the basis of Eq. (43), so this is one of the approaches). However, with inhomogeneities present in Eqs. (11) and (12), the Green function method is irreplaceable.

We illustrate the suggested considerations for the region of $x \ge l + vt$, $t \ge 0$, where the above-stated approach (on the basis of Eq. (63)) becomes especially effective. Let $T(x, t)$ be a solution of the problem:

$$\frac{\partial T}{\partial t} = a \frac{\partial^2 T}{\partial x^2}, \quad x > l + vt, \quad t > 0; \tag{64}$$

$$T(x, t)|_{t=0} = 0, \quad x \ge l; \tag{65}$$

$$\left(\beta_1 \frac{\partial T}{\partial x} + \beta_2 T\right)_{x=l+vt} = \beta_3 \varphi(t), \quad t \ge 0; \tag{66}$$

$$|T(x, t)| < +\infty, \quad x \ge l + vt, \quad t \ge 0, \tag{67}$$

where $\beta_1 = 0$, $\beta_2 = \beta_3 = 1$ in the case of the first boundary-value problem; $\beta_2 = 0$, $\beta_1 = \beta_3 = 1$ in the case of the second boundary-value problem; $\beta_1 = 1$, $\beta_2 = \beta_3 = -h$ (h is the relative coefficient of heat transfer) in the case of the third boundary-value problem. The function $T(x, t)$ is sought in the form of the generalized thermal potential of a simple layer using the curve $x = l + vt$

$$T(x, t) = \frac{\sqrt{a}}{2\sqrt{\pi}} \int_0^t \frac{\Psi(\tau)}{\sqrt{t - \tau}} \exp\left[-\frac{(x - l - v\tau)^2}{4a(t - \tau)}\right] d\tau, \tag{68}$$

where $\Psi(t)$ is the unknown potential density to be found from boundary condition (66). In the space of Laplace transforms expression (68) has the form

$$\bar{T}(x, p) = \frac{\sqrt{a}}{2\sqrt{p}} \exp\left(-\frac{x - l}{\sqrt{a}}\sqrt{p}\right) \bar{\Psi}\left(p - \frac{v}{\sqrt{a}}\sqrt{p}\right), \tag{69}$$

from which the operational form of the unknown density follows. Repeating the above reasoning, we find the following base relation for operational solution of boundary-value problem (64)-(67):

$$\overline{T}(x,p) = \overline{\Theta}(p)\left(1 - \frac{v/2a}{\sqrt{p/a}}\right)\exp\left(-\frac{x-l}{\sqrt{a}}\sqrt{p}\right)\overline{\varphi}\left(p - \frac{v}{\sqrt{a}}\sqrt{p}\right), \tag{70}$$

where

$$\overline{\Theta}(p) = \begin{cases} 1 & \text{for the first boundary-value problem;} \\ -1/\sqrt{p/a} & \text{for the second boundary-value problem;} \\ \dfrac{h}{h+\sqrt{p/a}} & \text{for the third boundary-value problem.} \end{cases}$$

Passing to the space of original functions by means of well-known rules of operational calculus [7], we obtain integral relations in a very compact form. As, for example, in the case of the first boundary-value problem:

$$T(x,t) = \frac{1}{2\sqrt{\pi a}}\int_0^t \frac{x-(l+v\tau)}{(t-\tau)^{3/2}}\varphi_1(\tau)\exp\left[-\frac{(x-l-v\tau)^2}{4a(t-\tau)}\right]d\tau.$$

In order to find an analytical solution of the inhomogeneous heat conduction equation with the inhomogeneous initial condition

$$\frac{\partial T}{\partial t} = a\frac{\partial^2 T}{\partial x^2} + f(x,t), \quad x > l + vt, \quad t > 0; \tag{71}$$

$$T(x,t)\big|_{t=0} = \Phi_0(x), \quad x \ge l, \tag{72}$$

and boundary conditions (66) and (67), it is necessary first to construct the corresponding Green function $G(x, t, x', \tau)$. By means of expression (70) we can do it with minimal calculations. Actually, according to Eq. (16), in the case of the third boundary-value problem for the function $q(x, t', x', \tau)$ we have:

$$\frac{\partial q}{\partial t'} = a\frac{\partial^2 q}{\partial x^2}, \quad x > l_0 + vt', \quad t' > 0; \tag{73}$$

$$q(x,t',x',\tau)\big|_{t'=0} = 0, \quad x > l_0; \tag{74}$$

$$\left.\frac{\partial q}{\partial x}\right|_{x=l_0+vt'} = h\left\{q\big|_{x=l_0+vt'} - \frac{(x'-l_0)-(v+2ah)t'}{4h\sqrt{\pi}(at')^{3/2}}\exp\left[-\frac{(l_0+vt'-x')^2}{4at'}\right]\right\}, \quad t' > 0; \tag{75}$$

$$|q(x,t',x',\tau)| < +\infty, \quad x \ge l_0 + vt', \quad t' \ge 0, \tag{76}$$

where $l_0 = l + v\tau$; $t' = t - \tau$. Now, in accordance with approach (70) we separate out the function $\varphi(t')$ in boundary condition (75):

$$\varphi(t') = \frac{(x'-l_0)-(v+2ah)t'}{4h\sqrt{\pi}(at')^{3/2}}\exp\left[-\frac{(l_0+vt'-x')^2}{4at'}\right],$$

find its representation in the form indicated in Eq. (70):

$$\bar{\varphi}\left(p - \frac{v}{\sqrt{a}}\sqrt{p}\right) = \frac{\sqrt{p} - \left(\frac{v}{\sqrt{a}} + h\sqrt{a}\right)}{2ah\left(\sqrt{p} - \frac{v}{2\sqrt{a}}\right)} \exp\left[-\frac{x' - l_0}{\sqrt{a}}\sqrt{p} + \frac{v}{a}(x' - l_0)\right]$$

and then, using Eq. (70), we pass to the representation for $\bar{q}(x, p, x', \tau)$

$$\bar{q} = \frac{1}{2\sqrt{a}} \frac{\sqrt{p} - \left(\frac{v}{\sqrt{a}} + h\sqrt{a}\right)}{\sqrt{p}\,(\sqrt{p} + h\sqrt{a})} \exp\left[-\frac{x + x' - 2l_0}{\sqrt{a}}\sqrt{p} + \frac{v}{a}(x' - l_0)\right]. \tag{77}$$

Resorting to the original functions in Eq. (77) and taking into account Eq. (16), we determine the Green function for the third boundary-value problem in the region of $x > l + vt$, $t > 0$:

$$G(x, t, x', \tau) = \frac{1}{2\sqrt{\pi a(t-\tau)}}\left\{\exp\left[-\frac{(x - x')^2}{4a(t-\tau)}\right] + \right.$$

$$\left. + \exp\left[-\frac{(x + x' - 2(l + v\tau))^2}{4a(t-\tau)} + \frac{v}{a}(x' - (l + v\tau))\right]\right\} -$$

$$- \left(h + \frac{v}{2a}\right) \exp\left\{[x + x' - 2(l + v\tau)]\,h + ah^2(t-\tau) + \frac{v}{a}[x' - (l + v\tau)]\right\} \times$$

$$\times\, \Phi^*\left[\frac{x + x' - 2(l + v\tau)}{2\sqrt{a(t-\tau)}} + h\sqrt{a(t-\tau)}\right], \tag{78}$$

where $\Phi^*(z) = 1 - \Phi(z)$; $\Phi(z) = (2/\sqrt{\pi}) \int_0^z \exp(-y^2)dy$ of the Laplace function. Assuming that in Eq. (77) $h = 0$, we find the Green function for the second boundary-value problem

$$G(x, t, x', \tau) = \frac{1}{2\sqrt{\pi a(t-\tau)}}\left\{\exp\left[-\frac{(x - x')^2}{4a(t-\tau)}\right] + \right.$$

$$\left. + \exp\left[-\frac{(x + x' - 2(l + v\tau))^2}{4a(t-\tau)} + \frac{v}{a}(x' - (l + v\tau))\right]\right\} -$$

$$- \frac{v}{2a} \exp\left[\frac{v}{a}(x' - (l + v\tau))\right] \Phi^*\left[\frac{x + x' - 2(l + v\tau)}{2\sqrt{a(t-\tau)}}\right]. \tag{79}$$

The limit passage with $(1/h) \to 0$ in Eq. (77) leads to the Green function for the first boundary-value problem

$$G(x, t, x', \tau) = \frac{1}{2\sqrt{\pi a(t-\tau)}}\left\{\exp\left[-\frac{(x - x')^2}{4a(t-\tau)}\right] - \right.$$

$$\left. - \exp\left[-\frac{(x + x' - 2(l + v\tau))^2}{4a(t-\tau)} + \frac{v}{a}(x' - (l + v\tau))\right]\right\}. \tag{80}$$

The integral representation of the analytical solution of problem (71), (72), (66), (67) has the form of

$$T(x,t) = \int_l^{\infty} \Phi_0(x') G(x,t,x',0)\, dx' +$$

$$+ a \int_0^t \left(\gamma_1 \frac{\partial G}{\partial x'} - \gamma_2 G \right)_{x'=l+v\tau} \varphi(\tau)\, d\tau + \int_0^t \int_{l+v\tau}^{\infty} f(x',\tau)\, G(x,t,x',\tau)\, d\tau dx' , \qquad (81)$$

where $\gamma_1 = 1$, $\gamma_2 = 0$ in the case of the first boundary-value problem; $\gamma_1 = 0$, $\gamma_2 = 1$ in the case of the second boundary-value problem; $\gamma_1 = 0$, $\gamma_2 = -h$ in the case of the third boundary-value problem.

For different boundary conditions the specific features of the method for the region of $\overline{\Omega}_t = \{x \in [0, y(t)],\ t \geq 0\}$ consist only in solving a finite difference equation and in passing to an original. This method can be extended to other regions and laws of boundary motion. Although the approach given here concerns boundary-value problems for Eqs. (64) and (71), it is also possible to consider equations of the following form

$$\frac{\partial T}{\partial t} = a\Delta T(M,t) - b^2 T(M,t) + \mathbf{v}\, \mathrm{grad}\, T(M,t) + F(M,t) , \qquad (82)$$

since by substitution

$$T(M,t) = U(M,t) \exp\left[-\frac{1}{2a} \mathbf{r} \cdot \mathbf{v} - \left(b^2 + \frac{1}{4a} \sum_{i=1}^{3} v_i^2 \right) t \right] , \qquad (83)$$

(here $M = M(x_1, x_2, x_3)$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ (v_i = const), b^2 = const, $\mathbf{r} = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$) Eq. (83) is reduced to the case

$$\frac{\partial U}{\partial t} = a\Delta U(M,t) + W(M,t) ,$$

where $W(M, t)$ is the new (known) function.

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